Theorem (Vitaly covering thm for Radon measures)

Let μ be a Radon measure on $\mathbb{R}^d$ and $A \subset \mathbb{R}^d$ & $\mathcal{B}$ is a family of closed balls s.t. $\forall x \in A$: $\inf\{r : B(x,r) \in \mathcal{B}\} = 0$. That is every point $x \in A$ is the center of an arbitrary small ball $B \in \mathcal{B}$. Then $\exists \{B_i\}_i$, $B_i \in \mathcal{B}$ disjoint balls s.t. $\mu(A \setminus \bigcup_i B_i) = 0$.

Proof We may assume that $\mu(A) > 0$. First we suppose that A is bounded. Using that μ is a regular measure: $\exists\, U \subset \mathbb{R}^d$ open set with

$$(8) \quad \mu(U) \leq \left(1 + \frac{1}{4Q(d)}\right) \mu(A) \;\&\; A \subset U$$

where $Q(d)$ is the constant from Besicovitch theorem.

W.L.G. we may assume that for $\forall B \in \mathcal{B}$, $B \subset U$. (We throw away the balls from $\mathcal{B}$ which are not contained in U)

Now we apply Besicovitch theorem for $\mathcal{B}$:

$A \subset \bigcup_{i=1}^{Q(d)} \bigcup_{B \in \mathcal{B}_i} B$. Hence $\exists i$: $\mu(A) \leq Q(d) \cdot \mu\left(\bigcup_{B \in \mathcal{B}_i} B\right)$

We can find a finite subfamily $\mathcal{B}_i' \subset \mathcal{B}_i$

s.t.

(9) $\boxed{\mu(A) \le 2Q(d)\,\mu\Big(\bigcup_{B\in\mathcal{B}_i'} B\Big)}$. Let $A_1 := A \setminus \bigcup_{B\in\mathcal{B}_i'} B$.

$$\mu(A_1) \le \mu\Big(\mathcal{U} \setminus \bigcup_{B\in\mathcal{B}_i'} B\Big) = \mu(\mathcal{U}) - \mu\Big(\bigcup_{B\in\mathcal{B}_i'} B\Big) \overset{(8),(9)}{\le} \Big(1+\frac{1}{4Q(d)} - \frac{1}{2Q(d)}\Big)\mu(A)$$

$$= u\cdot\mu(A) \text{ for } u := 1 + \frac{1}{4Q(d)} - \frac{1}{2Q(d)} < 1.$$

Using that $\mathcal{B}_i' \subset \mathcal{B}_i$ and the elements of $\mathcal{B}_i$ are disjoint balls we obtain that $\mathcal{B}_i'$ consists of disjoint, closed balls s.t.

$$\mu\Big(A \setminus \bigcup_{B\in\mathcal{B}_i'} B\Big) \le u\cdot\mu(A).$$

Then the proof is finished exactly as the Vitali Covering Thm for Leb. measures. That we choose an open set $\mathcal{U}_1 \subset \mathcal{U} \setminus \underbrace{\bigcup_{B\in\mathcal{B}_i'} B}_{\text{closed}}$ s.t.

(10) $\mu(\mathcal{U}_1) \le \Big(1+\frac{1}{4Q(d)}\Big)\mu(A_1)$ & $A_1 \subset \mathcal{U}_1$.

We throw away of any balls from $\mathcal{B}$ which is not contained in $\mathcal{U}_1$. The collection of balls is still denoted by $\mathcal{B}$. Then we repeat the steps above. This results a family $\mathcal{B}_{j_2}^{(2)}$ of finitely many disjoint balls s.t. for $A_2 := A_1 \setminus \bigcup_{B\in\mathcal{B}_{j_2}^{(2)}} B$ we have $\mu(A_2) \le u\cdot\mu(A)$.

Clearly $\bigcup_{B\in\mathcal{B}_{j_2}^{(2)}} B \cap \bigcup_{B\in\mathcal{B}_i'} B = \emptyset$ since $\bigcup_{B\in\mathcal{B}_{j_2}^{(2)}} B \subset \mathcal{U}_1$ & $\mathcal{U}_1 \cap \bigcup_{B\in\mathcal{B}_i'} B = \emptyset$.

Then we continue this & we construct $\mathcal{B}^{(n)}_{j_n}$ s.t.

$$\tilde{\mathcal{B}} := \mathcal{B}'_i \cup \bigcup_{n=2}^{\infty} \mathcal{B}^{(n)}_{j_n}$$ satisfies:

- $$\mu\Big(A \setminus \bigcup_{B\in\tilde{\mathcal{B}}} B\Big)=0$$
- If $B', B'' \in \tilde{\mathcal{B}}$, $B_1 \neq B_2$ then $B' \cap B'' = \emptyset$

Which completes the proof. $\blacksquare$

Now we prove an auxiliary theorem for Hausdorff measures

<u>Theorem A</u> (a) Let $E\subset\mathbb{R}^d$ be an arbitrary set, $s>0$. Then $\exists\, G\subset\mathbb{R}^d$ which is a G_δ-set (intersection of countably many open sets) s.t.

(11) $\quad E\subset G$ & $\mathcal{H}^s(G)=\mathcal{H}^s(E)$.

(b) If E is $\mathcal{H}^s$-measurable & $\mathcal{H}^s(E)<\infty$ then

(12) $$\exists F \ \underbrace{F_\sigma}_{\text{countable union of closed sets}}\text{-set},\ F\subset E \ \&\ \mathcal{H}^s(E\setminus F)=0.$$

<u>Proof (a)</u> If $\mathcal{H}^s(E)=\infty$ then $G=\mathbb{R}^d$ will do. Suppose that $\mathcal{H}^s(E)<\infty$. For every $i=1,2,\ldots$ we choose a $\frac{2}{i}$-cover $\{U_{i,j}\}_{j=1}^{\infty}$ satisfying: $U_{i,j}$ are open sets &

$$E\subset\bigcup_{j=1}^{\infty} E_{i,j}, \qquad |U_{i,j}|<\frac{2}{i},\ \forall i,j \ \& \ \sum_{j=1}^{\infty}|U_{ij}|^s<\mathcal{H}^s_{\frac{1}{i}}(E)+\frac{1}{i}$$

Let $G := \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \mathcal{U}_{i,j}$. Then G is a G_δ set, $E \subset G$.

Moreover

$$\mathcal{H}^s_{\frac{2}{i}}(G) \leq \mathcal{H}^s_{\frac{1}{i}}(E) + \frac{1}{i}. \Longrightarrow \mathcal{H}^s(E) = \mathcal{H}^s(G). \blacksquare$$

Using that every G_δ-set is a Borel set & Borel sets are $\mathcal{H}^s$-regular we get that $\mathcal{H}^s$ is a regular outer measure.

Proof (b) Let $E \subset \mathbb{R}^d$ be an $\mathcal{H}^s$-measurable set with $\mathcal{H}^s(E) < \infty$. We use part (a). It says that we can find open sets $\mathcal{O}_1, \mathcal{O}_2, \ldots$ s.t.

$$E \subset \mathcal{O}_1, \mathcal{O}_2, \ldots \;\&\; 0 = \mathcal{H}^s\left(\bigcap_{i=1}^{\infty} \mathcal{O}_i \setminus E\right) = \mathcal{H}^s\left(\bigcap_{i=1}^{\infty} \mathcal{O}_i\right) - \mathcal{H}^s(E).$$

For every i we can find closed sets $F_{i,j} \nearrow \mathcal{O}_i$ as $j \to \infty$, $\{F_{i,j}\}_{j=1}^{\infty}$ is an increasing sequence of sets, $\bigcup_{j=1}^{\infty} F_{i,j} = \mathcal{O}_i$

Hence

$$\lim_{j \to \infty} \mathcal{H}^s(E \cap F_{i,j}) = \mathcal{H}^s(E \cap \mathcal{O}_i) = \mathcal{H}^s(E).$$

That is for any $\varepsilon > 0$ $\exists \{j_i\}_i$ s.t.

$$\mathcal{H}^s(E \setminus F_{i j_i}) < \frac{\varepsilon}{2^i} \quad (i = 1, 2, \ldots)$$

$F_{i,j}$ is the region encircled by the red curve

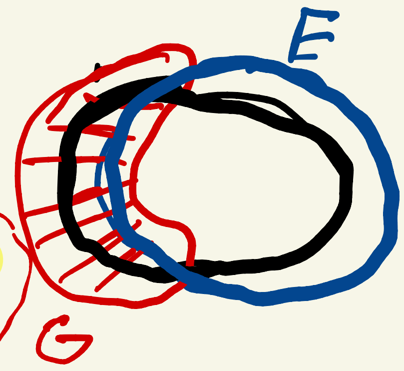
Let $F := \bigcap_{i=1}^{\infty} F_{i j_i}$. Then

(*)
$$\mathcal{H}^s(F) \geq \mathcal{H}^s(E \cap F) \geq \mathcal{H}^s(E) - \sum_{i=1}^{\infty} \mathcal{H}^s(E \setminus F_{i j_i}) \geq \mathcal{H}^s(E) - \varepsilon$$

We know that $\underset{\text{closed}}{F} \subset \bigcap_{i=1}^{\infty} \mathcal{O}_i$. Hence

$\mathcal{H}^s(F\setminus E) \le \mathcal{H}^s\left(\bigcap_{i=1}^{\infty} O_i \setminus E\right) = 0$. Now we use part (a) for the set $F\setminus E$. Namely, $\exists$ a G_δ set G s.t.

$F\setminus E \subset G$ & $\mathcal{H}^s(G) = 0$. Hence



$F\setminus G \subset E$ & $F\setminus G$ is an F_σ-set.

$\mathcal{H}^s(F\setminus G) \ge \mathcal{H}^s(F) - \mathcal{H}^s(G) > \mathcal{H}^s(E) - \varepsilon$. For each $\varepsilon = \frac{1}{2^n}$, $n\in\mathbb{N}$ we construct such an F_σ set like $F\setminus G$ above then they are all contained in E and their union is still an F_σ set which has the same measure as E. ■

In this way we have proven the following:

If E is $\mathcal{H}^s$-measurable & $\mathcal{H}^s(E) < \infty$ then $\mathcal{H}^s|_E$ is a Radon measure.

Lemma B Let $E \subset \mathbb{R}^d$ be an $\mathcal{H}^s$-measurable set with $\mathcal{H}^s(E) < \infty$ and let $\varepsilon > 0$. Then $\exists\ \delta = \delta(E,\varepsilon)$ s.t. for any collection of Borel sets $\{U_i\}_{i=1}^{\infty}$ with

$0 < |U_i| < \delta$ we have $\boxed{\mathcal{H}^s\left(E \cap \bigcup_i U_i\right) < \sum_i |U_i|^s + \varepsilon}$ (13)

Proof From the definition of $\mathcal{H}^s$, $\exists\ \delta > 0$ s.t.

$\mathcal{H}^s(E) < \sum_i |W_i|^s + \frac{\varepsilon}{2}$ holds for any δ-cover $\{W_i\}$ of E. Given Borel sets $\{U_i\}_{i=1}^{\infty}$ with

$0<|U_i|<\delta$. Then $\exists$ a δ-cover $\{V_i\}$ of $E\setminus\bigcup_i U_i$ s.t. (14) $\mathcal{H}^s(E\setminus\bigcup_{i=1}^{\infty}U_i)+\frac{1}{2}\varepsilon>\sum_{i=1}^{\infty}|V_i|^s.$

Note that $\{U_i\}_i\cup\{V_i\}_i$ is a δ-cover of E.

$$\mathcal{H}^s(E)<\sum_i|U_i|^s+\sum_i|V_i|^s+\frac{1}{2}\varepsilon \quad (15)$$

Putting (14) & (15) together:

$$\mathcal{H}^s(E\cap\bigcup_i U_i)=\mathcal{H}^s(E)-\mathcal{H}^s(E\setminus\bigcup_i U_i)$$

$$\leq\sum_i|U_i|^s+\sum_i|V_i|^s+\frac{1}{2}\varepsilon-\sum|V_i|^s+\frac{1}{2}\varepsilon=\sum_i|U_i|^s+\varepsilon$$

Vitali Covering Theorem for Hausdorff measures

Definition A collection of sets $\mathcal{V}$ is called a Vitali class for a set $E\subset\mathbb{R}^d$ if $\forall x\in E,\ \forall\delta>0\ \exists\, U\in\mathcal{V}$ with $x\in U$ & $0<|U|\leq\delta$, where $|U|$ is the diameter of the set U.

Theorem Let $E\subset\mathbb{R}^d$ be an $\mathcal{H}^s$-measurable set. Let $\mathcal{V}$ be a Vitali class of closed sets for E. Then

(a) We may select a countable disjoint sequence $\{U_i\}_i\subset\mathcal{V}$ s.t.

either $\sum_i|U_i|^s=\infty$ or $\mathcal{H}^s(E\setminus\bigcup_{i=1}^{\infty}U_i)=0$.

(b) If $\mathcal{H}^s(E)<\infty$ then given $\varepsilon>0$, we may require that

$$\mathcal{H}^s(E)\leq\sum_i|U_i|^s+\varepsilon.$$

Proof Fix $\varepsilon>0$. W.L.G. we may assume that $|U|<\varepsilon$ holds for all $U\in\mathcal{V}$. We select an arbitrary $U_1\in\mathcal{V}$. Assume that $U_1,\dots,U_m$ have already been chosen. Let $d_m := \sup\{|U| : U\in\mathcal{V},\ U\cap\bigcup_{i=1}^{m}U_i=\emptyset\}$.
If $d_m=0$ then $E\subset\bigcup_{1}^{m}U_i$. In this case part (a) holds. If $d_m>0$ then we choos U_{m+1} s.t.

$$U_{m+1}\cap\bigcup_{i=1}^{m}U_i=\emptyset \ \& \ |U_{m+1}|>\frac{1}{2}d_m.$$

Suppose that this process continuous ad ifinitum s.t. $\sum_{i=1}^{\infty}|U_i|^s<\infty$. If this does not hold that is $\sum_{i=1}^{\infty}|U_i|^s=\infty$ then we are ready with the proof of (a). For every we choos an arbitnary $x_i\in U_i$ and we define $B_i := B(x_i, 3|U_i|)$.

Claim $E\setminus\bigcup_{i=1}^{k}U_i\subset\bigcup_{i=k+1}^{\infty}B_i$.

Proof of the Claim Let $x\in E\setminus\bigcup_{i=1}^{k}U_i$. We can choose $U\in\mathcal{V}$ s.t. $x\in U$ & $U\cap\bigcup_{i=1}^{k}U_i=\emptyset$. This is so since the sets $\{U_i\}_{i=1}^{k}$ are closed & $\mathcal{V}$ is a Vitaly cover. Using that

$\sum_{i=1}^{\infty} |U_i| < \infty$ we get that $\lim_{i\to\infty} |U_i| = 0$.

So, we can find $m>k$ s.t. $2\cdot|U_m| < |U|$. Then $\exists\, j$ s.t. $k<j\le m$, $U\cap U_j \neq \emptyset$ & $|U| \le 2|U_j|$.

Namely, $|U_m| > d_m/2$. Hence $d_m < 2|U_m| < |U|$. So, by the definition of d_m we have $\exists\, j \le m$ s.t.

(16) $U \cap U_j \neq \emptyset$. However, $U \cap \bigcup_{i=1}^{k} U_i = \emptyset$ by assumption.

So, the j in (16) satisfies $k<j\le m$. From now on we assume that $k<j\le m$ is the smallest number for which (16) holds. Then we claim that

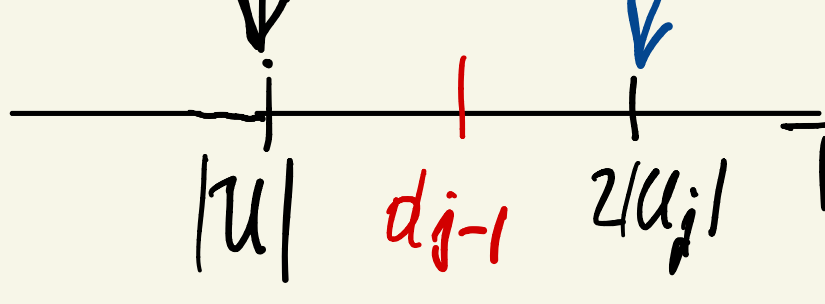
(17) $|U| < 2|U_j|$. Namely, $U \cap (U_1 \cup \dots \cup U_{j-1}) = \emptyset$ (18)

Since U_j is the smallest indexed element of $\{U_n\}$ having non-empty intersection with U.

Recall that $d_{j-1} = \sup\{|V| : V \in \mathcal{V},\ U \text{ satisfies } (18)\}$.

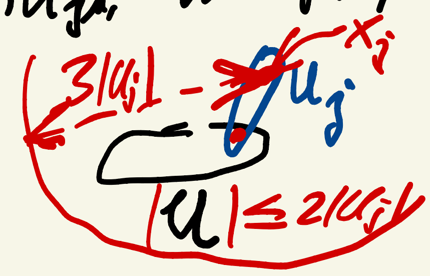
Hence, $|U| \le d_{j-1}$ (19) On the other hand, by the definition of U_j:

$$2|U_j| \ge d_{j-1}$$



This implies that (17) holds.

$|U| \le 2|U_j|;\ U \cap U_j \ne \emptyset$



Clearly, $U \subset B_j = B(x_j, 3|U_j|)$ so the Claim is proved.

Let $\delta > 0$ and let k be so large that $\forall i > k: |B_i| < \delta$. Then

$$\mathcal{H}^s_\delta\Big(E \setminus \bigcup_{i=1}^{\infty} U_i\Big) \le \mathcal{H}^s_\delta\Big(E \setminus \bigcup_{i=1}^{k} U_i\Big) \le \mathcal{H}^s_\delta\Big(E \cap \bigcup_{i=k+1}^{\infty} B_i\Big) \le$$

$$\overset{\text{Lemma B}}{\le} \sum_{i=k+1}^{\infty} |B_i|^s = 6^s \underbrace{\sum_{i=k+1}^{\infty} |U_i|^s}_{\to 0 \text{ as } k \to \infty}.$$

Hence

$\mathcal{H}^s_\delta\Big(E \setminus \bigcup_{i=1}^{\infty} U_i\Big) = 0$ holds for all $\delta > 0$. So

$\mathcal{H}^s\Big(E \setminus \bigcup_{i=1}^{\infty} U_i\Big) = 0$. This proves part (a).

<u>Proof of Part (b)</u> We may assume that $\varrho = \varrho(\varepsilon, m)$ is chosen as in the previous Lemma B.

If $\sum_{i=1}^{\infty} |U_i|^s = \infty$ then (b) is proved. Otherwise,

if $\sum_{i=1}^{\infty} |U_i|^s < \infty$ then by the previous Lemma B:

$$\mathcal{H}^s(E) = \mathcal{H}^s\Big(E \setminus \bigcup_{i=1}^{\infty} U_i\Big) + \mathcal{H}^s\Big(E \cap \bigcup_{i=1}^{\infty} U_i\Big)$$

$$= 0 + \mathcal{H}^s\Big(E \cap \bigcup_{i} U_i\Big) < \sum_{i=1}^{\infty} |U_i|^s + \varepsilon. \blacksquare$$